Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 1, pp. 112–117, 2009

DISCRETE-CONTINUUM MODEL OF SYMMETRIC SEPARATION OF A MATERIAL

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UDC 539.375

A problem of the beginning of motion of a finite-width cut in a linearly elastic plane under the action of symmetric external loading is formulated. The material on the way of cut propagation forms a layer (interaction layer). The stress–strain state of the material is postulated to be homogeneous across this layer. A system of integral boundary equations is obtained for determining the stress–strain state. Based on this system of equations, a discrete model of separation of the layer material is constructed under the assumption of a constant stress–strain state in an element of the interaction layer. The stress distribution in the pre-fracture zone is determined.

Key words: *characteristic size, integral boundary equation, linear elasticity.*

There are two basic approaches in modeling the formation of new material surfaces in fracture mechanics: fracture is considered as the motion of a mathematical cut in a solid or as the motion of a physical cut at a certain scale [1–5]. The model with the mathematical cut involves hypotheses of continuity for studying the neighborhood of the singular point with the use of various criteria [6]. If a certain scale is introduced, fracture is modeled as a discrete process [5, 7]. In this case, the solid is approximated by a set of structural elements interacting in accordance with laws chosen for this scale level. One substantial drawback of this approach is significant computational costs. Hence, it seems reasonable to develop approaches with the notions of mechanics of continuous media (fundamental solutions) for regions not subjected to fracture and a discrete description of the region being destroyed.

In solving the problem of separation within the framework of the discrete-continuum (semi-discrete) approach [7–10], let us consider the problem of the beginning of motion of a cut of width δ_0 [8] in a linearly elastic plane in accordance with the scheme corresponding to fracture of the tensile loading type (Fig. 1).

Let the separation trajectory correspond to straight-line motion of the cut in the direction that coincides with the direction of the OX_2 axis (see Fig. 1). The plane material bounded by the lines $x_1 = \pm\delta_0/2$ in the coordinates of the initial state forms an interaction layer with a uniform distribution of the stress–strain state across the layer. The quantity δ_0 characterizes the granular structure of the material in the case of inelastic deformation.

In contrast to the problem formulation in [8], the present statement takes into account the stress $\sigma_{22}(x_2)$ caused by shear stresses $\sigma_{21}(x_2)$ along the boundary with the half-plane, in addition to the stress $\sigma_{11}(x_2)$ in the layer.

We assume that the relation between the stresses and strains is described by expressions of the linear elasticity theory for the case of two-dimensional deformation. By virtue of problem symmetry, we consider only the upper half-plane ($x_1 \geqslant \delta_0/2$) (Fig. 2) and replace the action of the layer on the half-plane by the load onto this plane:

$$\boldsymbol{q}(x) = -(\hat{\sigma}_{11}\boldsymbol{e}_1 + \hat{\sigma}_{21}\boldsymbol{e}_2).$$

Here $x \equiv x_2/\delta_0$ is the dimensionless coordinate, $\hat{\sigma}_{ij} = \sigma_{ij}\beta$ $(i, j = 1, 2)$ are the dimensionless stresses, $\beta = 2(1-\nu^2)/(\pi E)$ is the material parameter, E is the modulus of elasticity, and ν is Poisson's ratio.

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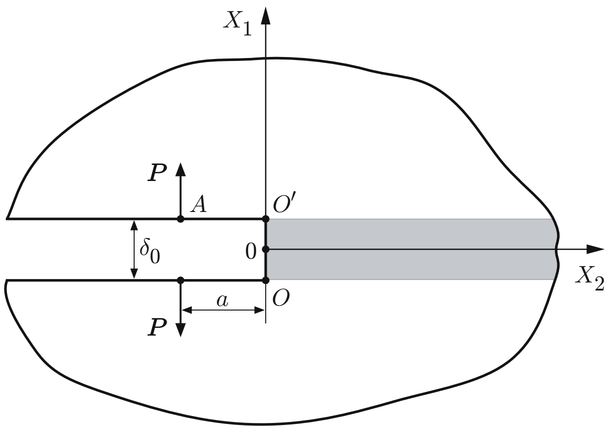

Fig. 1

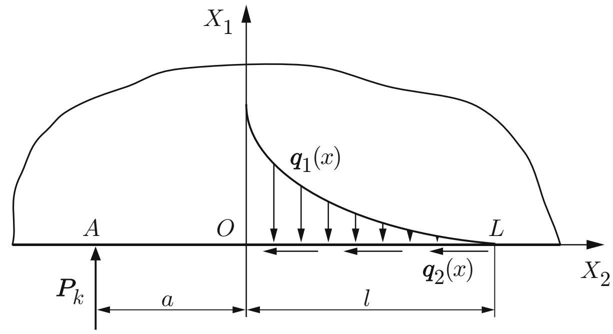

Fig. 2

Fig. 1. Scheme of fracture.

Fig. 2. Distribution of loads on the half-plane.

Flamant's relations [11] relate the external loads $\hat{\sigma}_{11}$ and $\hat{\sigma}_{12}$ to the boundary displacements by the dimensionless expressions

$$\hat{u}_1(x) = -\hat{P}\ln\left(\frac{x+a}{l+a}\right) + \int\limits_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi; \tag{1}$$

$$\hat{u}_2(x) = \int\limits_0^l \hat{\sigma}_{12}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi. \tag{2}$$

Here $\hat{u}_i = u_i/\delta_0$ $(i = 1, 2)$ are the dimensionless displacements, $\hat{P} = P\beta/\delta_0$ is the dimensionless force per unit thickness, and l is the distance between the origin and a distant point L with zero displacement.

We describe the behavior of the interaction layer material within the framework of the discrete model and present the material as a set of interacting δ_0-elements, which have a square planform. The main postulate of this model is the statement of a homogeneous stress–strain state in each element. By virtue of the homogeneity of the stress–strain state across the layer, the equilibrium condition implies that

$$\frac{\partial\hat{\sigma}_{22}}{\partial x} = -2\hat{\sigma}_{21}. \tag{3}$$

The displacements of the layer boundaries are determined from the conditions

$$\hat{u}_1(x) = \varepsilon_{11}(x)/2; \tag{4}$$

$$\hat{u}_2(x) = \int\limits_l^x \varepsilon_{22}(x)\,dx. \tag{5}$$

The stresses are related to the strains by Hooke's law as

$$\varepsilon_{11} = \hat{A}\hat{\sigma}_{11} - \hat{B}\hat{\sigma}_{22}; \tag{6}$$

$$\varepsilon_{22} = \hat{A}\hat{\sigma}_{22} - \hat{B}\hat{\sigma}_{11}, \tag{7}$$

where $\hat{A}$ and $\hat{B}$ are the dimensionless constants: $\hat{A} = (1-\nu^2)/(\beta E) = \pi/2$ and $\hat{B} = \nu(1+\nu)/(\beta E) = \nu\pi/(2(1-\nu))$.

Substituting expressions (4) and (6) into Eq. (1), we obtain an equation with respect to $\hat{\sigma}_{11}$ and $\hat{\sigma}_{22}$:

$$\frac{1}{2}\left(\hat{A}\hat{\sigma}_{11} - \hat{B}\hat{\sigma}_{22}\right) = -\hat{P}\ln\left(\frac{x+a}{l+a}\right) + \int\limits_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi. \tag{8}$$

Let us differentiate Eq. (2) with respect to x:

$$\varepsilon_{22} = \frac{d\hat{u}_2}{dx} = \int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi. \tag{9}$$

Substituting Eq. (7) into the left side of Eq. (9), we obtain

$$\hat{A}\hat{\sigma}_{22} - \hat{B}\hat{\sigma}_{11} = \int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi. \tag{10}$$

Let us eliminate $\hat{\sigma}_{22}$ from Eq. (8) by using Eq. (10). As a result, we obtain an equation with respect to $\hat{\sigma}_{11}$ and $\hat{\sigma}_{12}$:

$$(2\hat{A}^2 - \hat{B}^2)\hat{\sigma}_{11} = -2\hat{P}\hat{A}\ln\Big(\frac{x+a}{l+a}\Big) + 2\hat{A}\int_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi + \hat{B}\int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi. \tag{11}$$

Thus, we have a system of integral equations (10) and (11) supplemented by dependence (3). We write the resultant system of integrodifferential equations in the form

$$\hat{\sigma}_{11} + \lambda_1\int_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi + \lambda_2\int_0^l \hat{\sigma}_{12}(\xi)\ln\frac{1}{x-\xi}\,d\xi = \lambda_1\hat{P}\ln\Big(\frac{x+a}{l+a}\Big); \tag{12}$$

$$\hat{A}\hat{\sigma}_{22} - \hat{B}\hat{\sigma}_{11} - \int_0^l \hat{\sigma}_{12}\,\frac{1}{x-\xi}\,d\xi = 0; \tag{13}$$

$$\frac{\partial\hat{\sigma}_{22}}{\partial x} = -2\hat{\sigma}_{12}. \tag{14}$$

Here $\lambda_1 = -2\hat{A}/(2\hat{A}^2 - \hat{B}^2)$ and $\lambda_2 = -\hat{B}/(2\hat{A}^2 - \hat{B}^2)$.

As the stresses in each δ_0-element are constant, the generalized stresses in the ith element on the segment $i-1 \leqslant x \leqslant i$ are determined by the expressions

$$\sigma_{11}^{(i)} = \int_{i-1}^{i} \hat{\sigma}_{11}(x)\,dx, \qquad \sigma_{12}^{(i)} = \int_{i-1}^{i} \hat{\sigma}_{12}(x)\,dx, \qquad \sigma_{22}^{(i)} = \int_{i-1}^{i} \hat{\sigma}_{22}(x)\,dx.$$

Let us construct discrete expressions for integral operators in Eqs. (12) and (13). We consider the following operators:

$$A_1 = \int_0^l \hat{\sigma}_{11}(\xi)\ln\frac{|x-\xi|}{l-\xi}\,d\xi = \int_0^n \hat{\sigma}_{11}\ln\frac{|x-\xi|}{l-\xi}\,d\xi = \sum_{i=1}^{i=n}\int_{i-1}^{i}\hat{\sigma}_{11}\ln\frac{|x-\xi|}{l-\xi}\,d\xi$$

$$= \sum_{i=1}^{i=n}\sigma_{11}^{(i)}\Big(\int_{i-1}^{i}\ln|x-\xi|\,d\xi - \int_{i-1}^{i}\ln(l-\xi)\,d\xi\Big) = \sum_{i=1}^{i=n}\psi^{(i)}(x)\sigma_{11}^{(i)} - \sum_{i=1}^{i=n}C^{(i)}\sigma_{11}^{(i)}.$$

Here

$$\psi^{(i)}(x) = \int_{i-1}^{i}\ln|x-\xi|\,d\xi, \qquad C^{(i)} = \int_{i-1}^{i}\ln|l-\xi|\,d\xi,$$

$$A_2 = \int_0^l \hat{\sigma}_{12}(\xi)\,\frac{1}{x-\xi}\,d\xi = \int_0^n \hat{\sigma}_{12}\,\frac{1}{x-\xi}\,d\xi = \sum_{i=1}^{i=n}\psi_1^{(i)}(x)\sigma_{12}^{(i)}, \qquad \psi_1^{(i)}(x) = \int_{i-1}^{i}\frac{1}{x-\xi}\,d\xi.$$

Passing to discrete operators, we integrate the left and right sides of the considered system over the jth segment ($j-1 \leqslant x \leqslant j$). As a result, the operators A_1 and A_2 acquire the form

$$A_1^j[\sigma_{11}^{(i)}] = \sum_{i=1}^{i=n} \sigma_{11}^{(i)} \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \ln|x-\xi|\,d\xi\,dx - \sum_{i=1}^{i=n} \sigma_{11}^{(i)} \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \ln|l-\xi|\,d\xi\,dx,$$

$$A_2^j[\sigma_{12}^{(i)}] = \sum_{i=1}^{i=n} \sigma_{12}^{(i)} \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \frac{1}{x-\xi}\,d\xi\,dx.$$

We introduce the following notation:

$$\varphi^{(ji)} = \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \ln|x-\xi|\,d\xi\,dx, \quad \psi_1^{(ji)} = \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \frac{1}{x-\xi}\,d\xi\,dx, \quad C^{(ji)} = \int\limits_{j-1}^{j} \int\limits_{i-1}^{i} \ln(l-\xi)\,d\xi\,dx.$$

For $j > i$, we have

$$\varphi_{(+)}^{(ji)} = -(j-i)^2 \ln(j-i) + (1/2)(j-i-1)^2 \ln(j-i-1) + (1/2)(j-i+1)\ln(j-i+1) - 3/2,$$

$$\psi_{(+)}^{(ji)} = -2(j-i)\ln(j-i) + (j-i-1)\ln(j-i-1) + (j-i+1)\ln(j-i+1).$$

For $j = i$, we present the integral $\int\limits_{j-1}^{j} \int\limits_{j-1}^{j} \ln|x-\xi|\,d\xi\,dx$, $x \in [j-1; j]$ in the form

$$\int\limits_{j-1}^{j} \Big(\int\limits_{j-1}^{x} \ln(x-\xi)\,d\xi + \int\limits_{x}^{j} \ln(\xi - x)\,d\xi \Big)\,dx.$$

After integration, we obtain $\varphi_{(0)}^{(ji)} = -3/2$. For $\psi_1^{(ji)}$ with $j = i$, we have $\psi_1^{(ji)} = 0$.

For $j < i$, we have

$$\varphi_{(-)}^{(ji)} = (i-j)^2 \ln(i-j) - (1/2)(i-j+1)^2 \ln(i-j+1) - (1/2)(i-j-1)^2 \ln(i-j-1) - 1/2,$$

$$\psi_{(-)}^{(ji)} = 2(i-j)\ln(i-j) - (i-j-1)\ln(j-i-1) - (i-j+1)\ln(i-j+1).$$

For all i and j, we have

$$C^{(ji)} = -(n-i)\ln(n-i) + (n-i+1)\ln(n-i+1) - 1,$$

$$D^{(j)} = \int\limits_{j-1}^{j} \ln\frac{x+a}{l+a}\,dx = (j+a)\ln\Big(\frac{j+a}{n+a}\Big) - (j+a-1)\ln\Big(\frac{j+a-1}{n+a}\Big) - 1.$$

As $\sigma_{12} = \sigma_{12}^j$ on the segment $[j-1, j]$, Eq. (3) yields

$$\sigma_{22}^{(j)} - \sigma_{22}^{(j-1)} = -2\sigma_{12}^{(j)}.$$

Finally, with allowance for the notation used, the full discrete model of material separation by point forces acquires the form

$$\begin{gathered} \sigma_{11}^{(j)} + \lambda_1 A_1^j[\sigma_{11}^{(i)}] + \lambda_2 A_2^j[\sigma_{12}^{(i)}] = \lambda_1 \hat{P} D^{(j)}, \\ A\sigma_{22}^{(j)} - B\sigma_{11}^{(j)} - A_2^j[\sigma_{12}^{(i)}] = 0, \qquad \sigma_{22}^{(j)} - \sigma_{22}^{(j-1)} = -2\sigma_{12}^{(j)}. \end{gathered} \tag{15}$$

The discrete model (15) contains an infinite number of linear equations ($n \to \infty$) supplemented by the boundary condition on the end face $x = 0$: $\sigma_{22}^{(0)} = 0$. A finite number of elements, however, can be sufficient for a fairly accurate analysis of the pre-fracture region. Thus, the ratio $K = \sigma_{11}^{(1)}/\sigma_{11}^{(0)}$ for the first element is $K = 1.005$

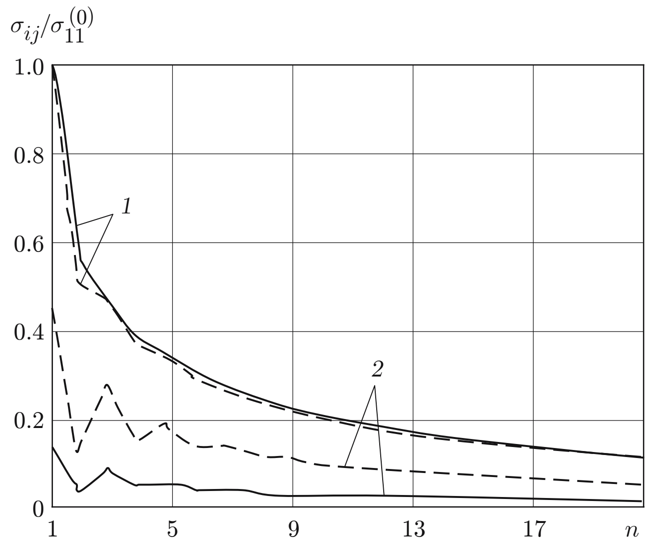

Fig. 3. Distributions of stresses σ_{11} (1) and σ_{22} (2) on the first 21 elements for $n = 1000$ and $a = 10$: the solid and dashed curves refer to $\nu = 0.15$ and 0.35, respectively.

for $n = 50$ and $n = 100$, $K = 1.003$ for $n = 100$ and $n = 500$, and $K = 1.000\,04$ for $n = 500$ and $n = 800$; for $n = 800$ and $n = 1000$, the value of K remains practically unchanged.

Figure 3 shows the stress distributions for the first 21 elements for $n = 1000$ and $a = 10$. It is seen that Poisson's ratio exerts practically no effect on the distribution of the stresses σ_{11} in the pre-fracture region, but significantly affects the stress σ_{22} and, correspondingly, the ratio of the maximum values of the stresses σ_{11} to σ_{22} in the interaction layer. In the numerical analysis [12] of extension of a plane with an elliptic cut and with the minimum radius of curvature tending to zero on the continuation of the major half-axis, the ratio σ_{11}/σ_{22} is a constant equal to 1/5. It follows from Fig. 3 that the stress σ_{22} is commensurable with the stress σ_{11} and can play a significant role in the formation of a plastic zone preceding the beginning of material separation. Moreover, the ratio σ_{11}/σ_{22} in the discrete model considered depends substantially on Poisson's ratio. Note that system (15) with $\nu = 0$ degenerates into the model proposed in [3]. In a more general case with $\nu \neq 0$, however, this system allows the distribution of the stress σ_{22} in the material located on the continuation of the cut in the linearly elastic plane to be taken into account.

This work was supported by the Russian Foundation for Basic Research (Grant Nos. 06-01-00047 and 07-01-96402).

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